Homogenous Spaces

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Definition 1 (Homogeneous space)**.** *Let M be a set with a Group G acting transitively on it, meaning there is a map $G \times M \to M$, $(g, x) \mapsto gx$ satisfying the properties:*

i) $(gh)x = g(hx)$ *for all* $g, h \in G$, $x \in M$*;*

ii) $ex = x$ *for all* $x \in M$ *with* e *being the neutral element of* G*;*

iii) for all $x, y \in M$ *there is a* $g \in G$ *such that* $y = gx$*.*

(M, G) *is called a homogeneous space.*

Example 1. $S^2 \subseteq \mathbb{R}^3$ *is a homogeneous space.*

Smooth manifolds are structures which are locally euclidian so we can perform Calculus on them. If G is a matrix group and $H \leq G$ a closed subgroup (therefore also a matrix group), the set of left cosets is defined by $G/H := \{gH : g \in G\}$, for which we can define the quotient map $\pi : G \to G/H$, $\pi(g) = gH$. We equip G/H with a topology, where $W \in G/H$ open $\Leftrightarrow \pi^{-1}(W) \subseteq G$ is open, and call it the quotient topology.
This can be generalized for G being a so called Lie Group, meaning it's a smooth manifold which is also a group with a smooth group operation.

Remark 1. *For an arbitrary surjection $q : X \to Y$ of a topological space X to a set Y, the resulting topology is called quotient topology with respect to q. q is actually continuous under this topology.*

Theorem 1. *If a matrix group G acts smoothly on a manifold M. If $x \in M$ has the stabiliser $Stab_G(x) \leq G$ and the orbit $Orb_G(x)$ is a closed submanifold, then $f : G/Stab_G(x) \to Orb_G(x)$, $f(gStab_G(x)) = gx$ is a diffeomorphism.*

Example 2. *For $n \geq 1$, $O(n)$ acts smoothly on $\mathbb{R}^n$. For any $v \in \mathbb{R} \setminus \{0\}$, $Orb_{O(n)}(v)$ is diffeomorphic to $O(n)/O(n-1)$.*

Projective Spaces

The unit group $\mathbb{R}^\times = \mathbb{R} \setminus \{0\}$ acts on $\mathbb{R}^{n+1}$ by the action $zx = xz^{-1}$. We denote the set of orbits as $\mathbb{R}P^n$: the n-dimensional projective space. Elements of $\mathbb{R}P^n$ are equivalence classes of the form $[x] = \{xz^{-1} : z \in \mathbb{R}^x\}$.

Remark 2. *$\mathbb{R}P^n$ can be identified with the set of all 1-dimensional subspaces of $\mathbb{R}^{n+1}$ and equipped with the quotient topology: $q_n : \mathbb{R}^{n+1} \setminus \{0\} \to \mathbb{R}P^n$, $q_n(x) = [x]$.*

Proposition 1. *$\mathbb{R}P^n$ is a smooth manifold, and it's quotient map q_n is smooth with a surjective derivative.*

Proposition 2. *We have the following diffeomorphisms:*

$$\mathbb{R}P^n \to O(n+1)/O(n) \times O(1), \quad \mathbb{R}P^n \to SO(n+1)/O(n);$$
$$\mathbb{C}P^n \to U(n+1)/U(n) \times U(1), \quad \mathbb{C}P^n \to SU(n+1)/U(n).$$

Definition 2 (Parabolic Subgroup)**.** *The parabolic subgroup $P \subseteq SL_2(\mathbb{C})$ is the set of all lower triangular matrices $\begin{pmatrix} u & 0 \\ w & v \end{pmatrix} \in SL_2(\mathbb{C})$*

Task: Prove, that $SL_2((C)/P \to \mathbb{C}P^1$ is a diffeomorphism.

Seminar Matrix Groups

Matrix exponentiation

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Definition 1 (convergence of series in $M_n(\mathbb{K})$). A series $\sum_{l=0}^{\infty} A_l = A_0 + A_1 + A_2 + \ldots of\ elements\ A_l \in M_n(\mathbb{K})$ converges if for each i, j the series $(A_0)_{ij} + (A_1)_{ij} + (A_2)_{ij} + \ldots$ converges to some $A_{ij} \in \mathbb{K}$.

Definition 2 (matrix exponentiation). $Exp(A) = \sum_{n=0}^{\infty} \frac{1}{n!} A^n$
$exp : M_n(\mathbb{K}) \to GL_n(\mathbb{K})$ defined as $exp(A) = Exp(A) \quad \forall A \in M_n(\mathbb{K})$

Lemma 1 (properties of the Exp).

1.) $Exp((u+v)A) = Exp(uA)Exp(vA)\, \forall u, v \in \mathbb{C}, \forall A \in M_n(\mathbb{K})$
2.) $Exp(A) \in GL_n(\mathbb{K})$ *and* $Exp(A)^{-1} = Exp(-A) \quad \forall A \in M_n(\mathbb{K})$
3.) $Exp(ABA^{-1}) = AExp(B)A^{-1} \quad \forall A, B \in M_n(\mathbb{K})$, *A invertible*
4.) $Exp(A+B) = Exp(A)Exp(B) \quad if\, A, B \in M_n(\mathbb{K})$ *commute*

Theorem 1 (Baker-Campbell-Hausdorff formula). $Exp(X)Exp(Y) = Exp(Z) \quad \forall X, Y \in M_n(\mathbb{K}) \quad with$
$Z = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]] - \frac{1}{12}[Y, [X, Y]] + ...$

Definition 3 (local inverse of exp). $Log(A) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} A^n \quad \forall ||A|| < 1$
$log : N_{M_n(\mathbb{K})}(I; 1) \to M_n(\mathbb{K})$ defined as $log(A) = log(A - I) \quad \forall \|A - I\| < 1$

Lemma 2 (local diffeomorphism). $exp : N_{M_n(\mathbb{K})}(0; log2) \to N_{GL_n(\mathbb{K})}(I; 1)$ *is a local diffeomorphism near 0 with local inverse log.*

Definition 4 (integral curve). A path $\alpha : \mathbb{R} \to \mathbb{R}^m$ is called an integral curve of a vector field $F : \mathbb{R}^m \to \mathbb{R}^m$ if $\alpha'(t) = F(\alpha(t))\, \forall t \in \mathbb{R}$.

Lemma 3. $A \in M_n(\mathbb{K})$, $\gamma(t) = e^{tA}\, \forall t \in \mathbb{R}$

i) $\forall X \in \mathbb{K}^n, \alpha(t) = X \cdot \gamma(t)$ *is an integral curve of the vector field on A*
ii) $\gamma(t)$ *is an integral curve of the vector field on* $M_n(\mathbb{K})$

Definition 5 (one-parameter group). A one-parameter group in G is a continuous function $\gamma : \mathbb{R} \to G$ which is differentiable at 0 and also satisfies $\gamma(s + t) = \gamma(s)\gamma(t)\, \forall s, t \in \mathbb{R}$.

Theorem 2 (matrix valued differential equations). *For* $A, C \in M_n(\mathbb{R})$ *with* $A \neq 0$ *the differential equation* $\begin{cases} \alpha'(t) = \alpha(t) \cdot A \\ \alpha(0) = C \end{cases}$ *has a unique solution* $\alpha : \mathbb{R} \to M_n(\mathbb{R})$ *defined as* $\alpha(t) = Ce^{tA}$

Excercise 1. Compute the matrix exponential $Exp(tA)\, \forall t \in \mathbb{R}\, for\ A = \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$